AN AREA EXPLORATION

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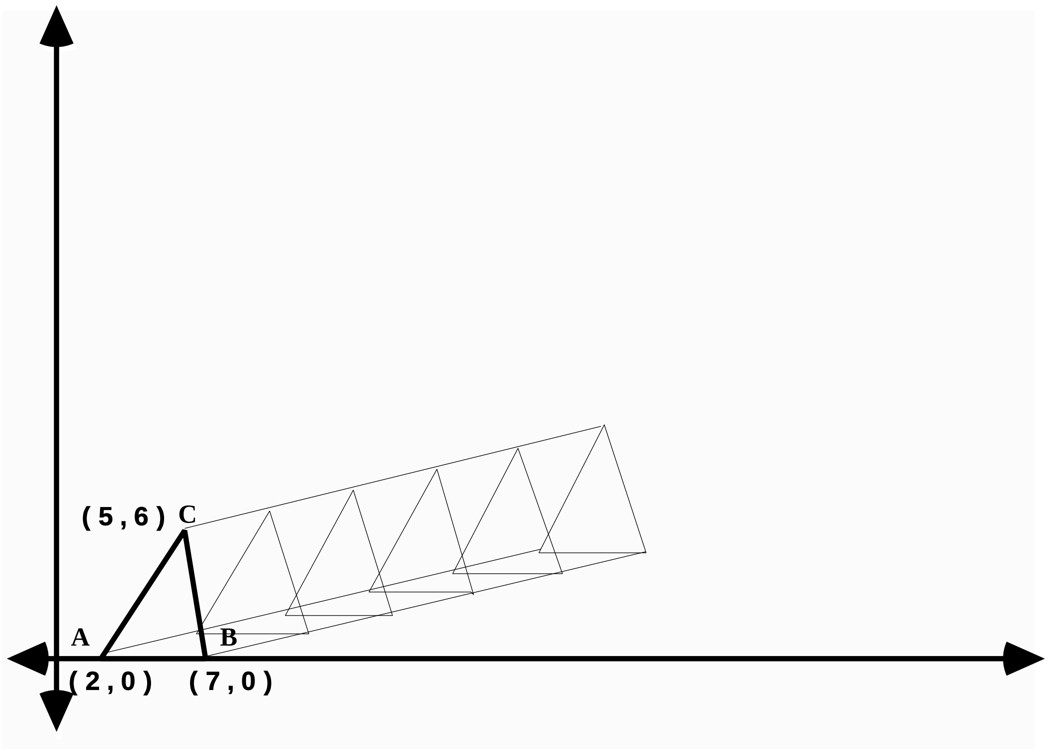
Sketched and labeled on the Cartesian graph below is ΔABC with vertices A(2,0), B(7,0), and C(5,6). This triangle can be written using matrix notation as :

$$\begin{matrix} A & B & C \end{matrix}$$
$$\begin{bmatrix} 2 & 7 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$



The lighter lines show the "movement" of ΔABC as it is translated **4t** spaces to the right and **t** spaces up for $\mathbf{0 \le t \le 5}$. While the triangle is translated continuously along a path, the lighter triangles, representing the positions of the translated triangle for **t = 1, 2, 3, 4,** and **5**, may help you see how it travels.

This translation could be indicated using matrix addition in the following way:

$$\begin{bmatrix} 2 & 7 & 5 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 4t & 4t & 4t \\ t & t & t \end{bmatrix}$$

When **t = 1**: $\begin{bmatrix} 2 & 7 & 5 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 4 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 11 & 9 \\ 1 & 1 & 7 \end{bmatrix}$

On the graph above, the first light triangle has coordinates (6,1), (11,1), and (9,7).

When **t = 2**: $\begin{bmatrix} 2 & 7 & 5 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 8 & 8 & 8 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 15 & 13 \\ 2 & 2 & 8 \end{bmatrix}$

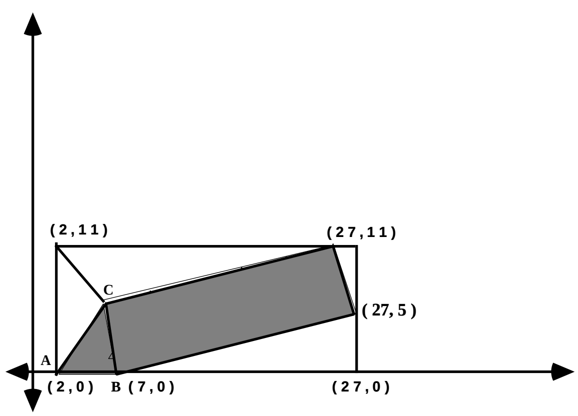
When **t** finally takes on the value of **5**:

$$\begin{bmatrix} 2 & 7 & 5 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 20 & 20 & 20 \\ 5 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 27 & 25 \\ 5 & 5 & 11 \end{bmatrix}$$

DETERMINING AN AREA

As ΔABC is translated along its defined path, it "sweeps out" a 2-dimensional figure. That figure is actually a pentagon. By using *encasement*, the *shoelace method* (see IMSA Math Journal, Vol. II, # 1, Fall 1993), or a geometric approach it can be verified that the area of this figure is 145 square units.

by Encasement



$$\text{Area} = 25 \cdot 11 - \frac{1}{2} \cdot 20 \cdot 5 - \frac{1}{2} \cdot 2 \cdot 6 - \frac{1}{2} \cdot 23 \cdot 5 - \frac{1}{2} \cdot 3 \cdot 11$$

Area = 145 square units

by The Shoelace Method

$$\frac{1}{2} \ \textbf{abs} \begin{bmatrix} 2 & 0 \\ 7 & 0 \\ 27 & 5 \\ 25 & 11 \\ 5 & 6 \\ 2 & 0 \end{bmatrix} =$$

$$\frac{1}{2} \left| (0+35+297+150+0) - (0+125+5+12) \right| =$$

$$\frac{1}{2} \cdot 290$$

Area = 145 square units

You might notice that the area of the pentagon is the sum of the areas of the original ΔABC and the quadrilateral which has been "added onto" ΔABC. The quadrilateral looks very much like a rectangle. It is not. This can be verified by comparing the slope of line segment BC and the slope of the line which extends from vertex B to the point (27,5). The figure is actually a parallelogram so extra effort will be needed to find its height and compute its area geometrically.

AN EXTRA BONUS

If we try to describe the path of the vertex (7,0) as it is translated, we see that it is clearly a line segment. The slope of the line, $\frac{1}{4}$, is defined by the ratio $\frac{t}{4t}$.

We can write this line segment as $\begin{cases} x = 7 + 4t \\ y = 0 + t \end{cases}$ for $0 \le t \le 5$.

These equations define the line segment *parametrically*. Try graphing them on a graphics calculator in parametric mode. The window will need to be set so that $0 \le t \le 5$. What window settings for x and y are necessary?

It is possible to eliminate the parameter, t:

$$y = t \text{ so substituting } y \text{ for } t \text{ yields } x = 7 + 4y$$
$$\text{or } y = \frac{1}{4}x - \frac{7}{4}.$$

Graph this line on your graphics calculator in function mode. It contains the line segment described earlier. To graph only the segment, graph

$$Y_1 = \left(\frac{1}{4}x - \frac{7}{4}\right) \cdot (7 \le x) \cdot (x \le 27)$$

(The ≥ and ≤ are found under the TEST menu of a TI-81 or 82.
The window must be set large enough. Dot mode is preferable)

As a last check, substitute the points (7, 0) and (27, 5) into the equation to verify that they do, indeed, lie on this line. They should, in fact, be the endpoints of the desired line segment.