Comprehensive Course Syllabus

MAT421 – Number Theory

Course Description:

Number Theory re-acquaints students to the number system they have been familiar with since they first learned 1, 2, 3. Students learn how the number system really fits together and how to exploit this structure to discover new relationships among the numbers and new applications for these relationships. The course serves as a stepping stone to higher mathematics by introducing students to rigorous proof and deep abstraction while remaining in mostly familiar territory. Curious students who want to get a glimpse of the powerful techniques and get a taste of higher mathematics but who are unsure of their readiness to handle the full abstraction of the more difficult pieces of material may elect a pass/fail option.

INSTRUCTOR(S):

- Name(s): Fall 2019: Micah Fogel
- Office Number(s): A157 (Math office)
- Office hours: daily, 8AM – 10AM
- Telephone number(s): Fall 2016: x5086
- Email address(es): fogel@imsa.edu

Meeting Days, Time and Room(s)
ABCD days, mod 3, Room A151

Text(s) / Materials:

No textbook is being used this semester. "Number Theory and its Applications" 6th ed. by Rosen is a reference for the course. Students are encouraged to find online or print text to supplement the material from the course lectures and discussions.

The primary source material for the course is the lectures and discussions which occur in the classroom, and students will be expected to keep careful notes.

Additional reading materials may be distributed by the teacher to emphasize particular topics.

Essential Content:

The following mathematics department standards are *assessed* in this course:

- A. Students studying mathematics at IMSA demonstrate a disposition and propensity to use mathematics, a variety of problem solving strategies, and creative thought to solve problems by:
  - A.1. investigating and gaining insight into mathematical concepts by selecting and using a variety of traditional and creative problem solving strategies and methodologies.
  - A.2. posing, solving, and extending both multi-step routine and multi-step unconventional problems.
  - A.3. interpreting, generalizing , and verifying the understanding gained in the problem solving process and extending it to new settings.
  - A.4. using a variety of resources and problem solving approaches.
- B. Students studying mathematics at IMSA reason logically in mathematical situations and understand the nature, role, and necessity of proof and counterexample in mathematical reasoning by:
  - B.1. demonstrating understanding of an axiomatic system.
  - B.2. reasoning inductively and deductively.
  - B.3. making and testing conjectures, creating proofs, and identifying counterexamples.
  - B.5. analyzing and critiquing proofs created by themselves and others.
- C. Students studying mathematics at IMSA communicate clearly and accurately about mathematical relationships and results by:
  - C.3. presenting mathematical work and results using the power of mathematical language effectively.
- G. Students studying mathematics at IMSA understand the underlying concepts and characteristics of mathematical functions and relations by:
  - G.3. evaluating and manipulating functions, creating multiple representations for a single function.

In addition, many more of the IMSA mathematics standards are *addressed* by the course, and students are expected to practice and improve along these dimensions:

- A.5. demonstrating confidence, persistence, and reflective analysis of the effectiveness of an approach when attempting to solve a problem.
- B.4. enhancing inductive and deductive reasoning through the use of intuition, imagination, and other forms of reasoning.
- C.2. accurately recording and effectively communicating using proper notation, vocabulary, and usage in a variety of modalities (written, oral, graphic, algebraic, etc.)
- E. Students studying mathematics at IMSA understand and employ the power, economy, clarity, and elegance of mathematical representations by:
  - E.1. recognizing that mathematical representations carry specific meanings and using mathematical notation correctly to enhance clarity and avoid ambiguity.

    - E.3. recognizing the structure underlying a mathematical representation and utilizing this structure in analysis and problem solving.
- L. Students studying mathematics at IMSA use technology to gain insight and obtain different perspectives on problems by:
    - L.2. using technology to facilitate doing, exploring, and understanding of mathematics.

SSLs and Outcomes:

The primary SSL's addressed in this course are:

- I. Developing the Tools of Thought
    - I.B Construct questions which further understanding, forge connections, and deepen meaning.
    - I.D Evaluate the soundness and relevance of information and reasoning.
    - III.A Use appropriate technologies as extensions of the mind.
    - III.C Recreate the beautiful conceptions that give coherence to structures of thought.

Instructional Design and Approach:

The course is problem-centered, as the need or desire to explain patterns discovered by simply "playing around" with numbers leads to a sequence of increasingly complex problems that have to be solved. The course is inquiry-based to a small extent, as the teacher strongly directs the lines of inquiry at first since the students don't have enough experience in the area to ask the right kinds of inquiry questions for the first few weeks of the course. As students gain experience, the direction of the course can change based upon the questions that the students ask. Toward the end of the course, there are several mini-units which are chosen by students, where students see application of the core material to problems in the areas that most interest them.

Student Expectations:

Students in Number Theory are expected to:

- Complete a large number of homework assignments in a timely manner. Homework is assigned to help students learn material in time for it to be applicable to the next section of material. Thus late homework exacts an intellectual penalty in not being prepared. It is also given a grade penalty to reflect its lateness.
- Keep current with course material. This material builds on itself, and not having complete understanding of today's material will make it much more difficult to learn tomorrow's.
- Keep good course notes. Not all material covered is in the textbook.
- Practice with problems beyond the assigned homework, if they expect to earn high grades.
- Participate in class discussions.
- Seek help early if concepts are not clear.

Assessment Practices, Procedures, and Processes:

There are frequent homework assignments which must be completed in a timely manner. To ensure students are keeping up with material, homework accounts for a significant portion of the grade.

Shorter quizzes are also fairly frequent, every two or three weeks. Quizzes are divided into basic computations that students must know how to do, more difficult conceptual questions that students should attempt if they want to reach the "next stage" in their mathematical development, and difficult theoretical questions and proofs which will challenge top students. Completing the basics and starting the conceptual will earn the student a passing grade; a "B" student should get the bulk of the concepts and be able to start the theory, while students trying to earn an "A" grade must make significant inroads on the theory.

There is a final exam for the course, required for students taking the course for a grade but optional for students on the pass/fail option who have been working at at least a "B" level throughout the semester.

Grades for the quarters are based on a 2/3 weighting of quiz and test scores with 1/3 weighting on homework scores. The semester grade is based upon 20% final exam score (if the student elects or is required to take the final exam) with the remaining 80% split in the same 2/3 – 1/3 relationship between tests and homework. If the final exam is not taken, the grade is simply the current term grade when the course ends.

Sequence of Topics and Activities

- Axiomatic treatment of the integers (approx. 2 weeks)
  - Axioms for equals, addition, multiplication, and ordering
  - Looking at basic principles that aren't as basic as you thought they were!
  - Defining multiplication in terms of addition
  - Well-ordering and induction

- Divisibility and factorization (aprox 2 weeks)
    - Division algorithm
    - Primes and factoring
    - GCD and Euclidean algorithm
    - Fundamental Theorem of Arithmetic
    - Linear Diophantine equations
- Multiplicative Functions (approx 2-3 days)
    - Counting and adding factors
    - Perfect numbers
- Modular arithmetic (approx 1½ weeks)
    - Mechanics of modular arithmetic
    - Manipulation of congruence equations
    - Chinese Remainder Theorem
- Euler's totient function and reduced residue systems (approx 1 week)
    - Reduced residue systems
    - Euler's function and its properties
    - Theorems of Fermat, Euler, and Wilson
    - Introduction to pseudoprimes
- Order and primitive roots (approx 1½ weeks)
    - Powers and polynomials in modular arithmetic
    - Roots of unity
    - Order and primitive roots
- Rational and real numbers (approx 1 – 1½ weeks)
    - Constructing the rational and real numbers, and proving their properties
    - Investigating the nature of positional notation systems, especially as applied to rational numbers
    - Cardinality

This material is the core of the course and occupies roughly 2/3 of the time available. The remaining time is devoted to "mini-units" chosen by the students. Some of the mini-units that have been studied in the recent past include:

- Codes, including cryptography and error detecting/correcting codes
- Quadratic reciprocity
- Continued fractions
- Advanced primality tests and factorization methods
- Sums of squares
- Combinatorial Game Theory and Surreal Numbers
- Peano axioms and constructing the real numbers "from scratch"